Quantum Field Theory – a primer

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1 Introduction

This text on QFT is strongly biased as far as the selected material is concerned. It was mainly written in order to provide, in a brief survey form, some background necessary to start a thesis work in my group.

The text is also quite "orthogonal" to many modern text books on QFT. The latter tend to take the Gell-Mann–Low formula (or an equivalent path integral formula) as a starting point, or at least arrive at it along sometimes quite different lines than I do, based mainly on classical analogies. They then concentrate on the identification of the appropriate interaction Lagrangean for the Standard Model, and on the nontrivial delicacies of computation of Feynman diagrams.

I shall rather concentrate on the conceptual structure of QFT with emphasis on its features as a genuine quantum theory: Hilbert space, expectation values, fluctuations, spectral properties, etc., and essentially stop at the GML formula (Sect. 2.8). I then turn to conformal quantum field theory, both in 4 and in 2 spacetime dimensions, which is my personal special field of research.

1.1 Beyond QM

Quantum Mechanics cannot be the ultimate theory of subatomic particle interactions for several reasons:

– QM is a non-relativistic theory (the Schrödinger equation is a transcription of the non-relativistic law $E_{\rm kin} = p^2/2m$ plus the deBroglie relations).

– QM assumes a fixed number of particles, therefore cannot treat particle production processes (which occur in collider experiments).

– QM cannot, in particular, properly treat light (a) which is necessarily relativistic and (b) with photons continuously being produced and annihilated (radiation of matter, thermodynamics with photon number $N(T) \sim U(T)/\overline{E} \sim T^4/T = T^3$.

– QM started from the observation that light and matter have the same corpuscular *and* wave features. However, the wave function is a *probability amplitude* which is conceptually very different from the electromagnetic *field amplitude*: the former is part of the description of a *state*, while the latter is an observable on the same footing as observables like momentum = *operators* in Schrödinger QM.

1.2 QM particles vs. quantum fields

The dynamical variables of a classical particle are its position and momentum. In QM, they become Hilbert space operators X_i and P_j, subject to canonical CRs.

The position is *not* an observable of a classical field, but instead the field amplitude $\phi(x)$ at a given position x is an observable. Correspondingly, the field amplitudes become operators $\phi(x)$ in QFT, with x just a label. There is no position operator X in QFT. On the other hand, classical fields possess energy and momentum densities, which typically are some polynomials function of the field and its gradient. The corresponding polynomials of the quantum field will be the energy and momentum density quantum fields, and their integrals are the Hamiltonian H and the momentum operator P.

2 Quantum fields

2.1 Relativistic symmetries

In the Heisenberg picture of QM, observables are functions of time, evolving according to

$$U(t)A(t_0)U(t)^* = A(t_0+t)$$

with the unitary time evolution operator $U(t) = e^{iHt}$, where H the Hamilton operator. The infinitesimal version of the time evolution is the Heisenberg equation

$$\partial_t A(t) = i[H, A(t)].$$

The fact that the time evolution can be written in this form ($H = \text{const}$), and that U is unitary, reflects the symmetry of the system under time translations.

Quantum fields depend on space and time, and a relativistic quantum system is invariant under space- and time-translations. Therefore one has

$$U(a)\phi(x)U(a)^* = \phi(x+a)$$

where $x = (x^0 = ct, \vec{x})$ and $a = (a^0, \vec{a})$, and

$$U(a) = e^{iP_\mu a^\mu} \equiv e^{i(P^0 a^0 - \vec{P}\cdot\vec{a})}.$$

$cP^0 = H$ is the Hamiltonian, and $\vec{P}$ are the components of the momentum. The infinitesimal version is

$$i[P_\mu, \phi(x)] = \partial_\mu \phi(x).$$

Symmetry under Lorentz transformations is implemented in the same way:

$$\phi(\Lambda x) = U(\Lambda)\phi(x)U(\Lambda)^*,$$

infinitesimal:

$$i[M_{\mu\nu}, \phi(x)] = (x_\mu\partial_\nu - x_\nu\partial_\mu)\phi(x),$$

where $U(\Lambda) = e^{i\lambda^{\mu\nu}M_{\mu\nu}}$. In particular, $L_i = \frac{1}{2}\varepsilon_{ijk}M_{jk}$ is the orbital momentum of the field. For a boost in direction i by the velocity $v = c\cdot\tanh\theta$ (i.e., $(\Lambda x)^0 = \cosh\theta\cdot x^0 + \sinh\theta\cdot x^i$ etc.), one has $\lambda^{0i} = -\lambda^{i0} = \frac{1}{2}\theta$ (all others = zero).

The composition laws for translations and Lorentz transformations defines the Poincaré group: Let $U(a,\Lambda) = U(a)U(\Lambda)$. Then

$$U(a_1,\Lambda_1)U(a_2,\Lambda_2) = U(a_1+\Lambda_1 a_2, \Lambda_1\Lambda_2).$$

The infinitesimal version is the Lie algebra

$$[P_\mu, P_\nu] = 0, \quad [M_{\mu\nu}, P_\kappa] = i(\eta_{\nu\kappa}P_\mu - \eta_{\mu\kappa}P_\nu),$$

$$[M_{\mu\nu}, M_{\kappa\lambda}] = i(\eta_{\nu\kappa}M_{\mu\lambda} - \eta_{\mu\kappa}M_{\nu\lambda} - \eta_{\nu\lambda}M_{\mu\kappa} + \eta_{\mu\lambda}M_{\nu\kappa}).$$

Lorentz transformation laws for vector fields (like the electromagnetic vector potential), for tensor fields (like the electromagnetic field strengths), or for spinor fields (like the Dirac field for the electron) are more complicated:

$$i[M_{\mu\nu}, \phi_a(x)] = \big((x_\mu\partial_\nu - x_\nu\partial_\mu)\delta^b_a + (\sigma_{\mu\nu})^b_a\big)\phi_b(x),$$

where the fields have several components and the tensors $(\sigma_{\mu\nu})^b_a$ constitute different matrix representations of the Lie algebra of the Lorentz group.

We assume that there is a unique state vector $\Omega \in \mathcal{H}$ which is invariant under U, in particular, $P_\mu\Omega = 0$. This state has zero energy and momentum, and is called the vacuum state. All other states should have positive energy, i.e., the vacuum is the ground state.

2.2 Causality and Locality

Special relativity includes, apart from the Poincaré symmetry, another new feature: Causality. Since causal effects should always occur after their cause, two events at spacelike separation cannot have any influence on each other: namely, in a suitable Lorentz frame, either event would be after the other.

In quantum theory, typical "events" are measurements performed at some point in space and time. Absence of influence means that the corresponding quantum operators commute. In quantum field theory, the typical quantum operators at the point x in space-time are field strengths $\phi(x)$. Therefore, we should have that

$$[\phi(x), \phi(y)] = 0 \quad \text{if} \quad (x-y)^2 < 0.$$

This is also referred to as Local Commutativity, or Locality, for short.

2.3 Particles

The notion of "particle" has never entered the previous characterization of quantum fields. How are quantum fields associated with particles?

We have to find Hilbert space realizations of the above (and possibly more) algebraic relations among operators. The "particles" arise as special states in the

Hilbert space which "behave like" multi-particle states, in the sense of (asymptotically well-localized) distributions of energy and momentum. An example of what this means, will be presented in the next subsection on free fields.

Notice here that the Hilbert space realization requires, among other things, a unitary representation of the Poincaré group:

Wigner has identified "particles" with irreducible representations of the Poincaré group. One-particle states form irreducible representations of the Poincaré group: in the simplest (scalar) case, one has an (improper) basis of momentum eigenstates $|k\rangle$ for four-momentum vectors on the "mass shell"

$$k \in H_m = \{k \in \mathbb{R}^4 : \ k_\mu k^\mu = m^2, \ k^0 > 0\}$$

(we henceforth put $c = \hbar = 1$), such that

$$P_\mu |k\rangle = k_\mu |k\rangle \qquad \Leftrightarrow \qquad U(a)|k\rangle = e^{iak}|k\rangle = e^{i(\omega(\vec{k})t - \vec{k}\vec{a})}|k\rangle,$$

where $\omega(\vec{k}) = \sqrt{\vec{k}^2 + m^2}$ is the energy belonging to the momentum $\vec{k}$, and

$$U(\Lambda)|k\rangle = |\Lambda k\rangle.$$

For spinor and tensor fields, one-particle states arise in multiplets for each momentum. The law under translations is the same, but under Lorentz transformations, there is an additional matrix representation mixing the components of the spinor or tensor multiplets.

The full Hilbert space decomposes into a subspace of one-particle states, plus an orthogonal complement. States in the complement evolve, asymptotically at large times, into superpositions of states that "look like" multi-particle states (e.g., the expectation value of the energy density at $t \to \pm\infty$ is well-localized along the trajectories of the particles). This is a highly non-trivial dynamical feature ("asymptotic completeness"), and the multi-particle state content of the Hilbert space is in general unknown. Only in interaction-free QFT, multi-particle states look like multi-particle states at all times, and the Hilbert space is just the Fock space over the one-particle subspace (see below).

2.4 Free fields

2.4.1 CCR

We assume the existence of operators $a(k)$ and their adjoints $a^*(k)$ for each momentum k on the mass-shell H_m, satisfying the momentum space canonical commutation relations (CCR)

$$[a(k), a(k')] = 0, \qquad [a(k), a^*(k')] = (2\pi)^3 \cdot 2\omega(\vec{k}) \cdot \delta(\vec{k} - \vec{k}').$$

From these, we can construct the hermitean "time-zero" fields

$$\phi(\vec{x}) := (2\pi)^{-3} \int \frac{d^3k}{2\omega(\vec{k})} \big(a(k)e^{i\vec{k}\cdot\vec{x}} + a^*(k)e^{-i\vec{k}\cdot\vec{x}}\big),$$

$$\pi(\vec{x}) := (2\pi)^{-3} \int \frac{d^3k}{2i} \big(a(k)e^{i\vec{k}\cdot\vec{x}} - a^*(k)e^{-i\vec{k}\cdot\vec{x}}\big).$$

These satisfy the position space CCR

$$[\phi(\vec{x}), \phi(\vec{x}')] = [\pi(\vec{x}), \pi(\vec{x}')] = 0, \qquad [\phi(\vec{x}), \pi(\vec{x}')] = i\delta(\vec{x} - \vec{x}'),$$

which are regarded as the quantum version of classical canonical Poisson brackets of a field and its conjugate momentum.

We now solve the equation of motion

$$(\Box + m^2)\varphi(t, \vec{x}) = 0$$

with initial values

$$\varphi(0, \vec{x}) = \phi(\vec{x}), \qquad \partial_t\varphi(0, \vec{x}) = \pi(\vec{x}).$$

The unique solution is

$$\varphi(t, \vec{x}) := (2\pi)^{-3} \int \frac{d^3k}{2\omega(\vec{k})} \big(a(k)e^{i(\vec{k}\cdot\vec{x}-\omega(\vec{k}))} + a^*(k)e^{-i(\vec{k}\cdot\vec{x}-\omega(\vec{k}))}\big).$$

This can be written in a covariant form as

$$\varphi(x) = \int_{H_m} \widetilde{dk} \big(a(k)e^{-ik_\mu x^\mu} + a^*(k)e^{ik_\mu x^\mu}\big),$$

where

$$\widetilde{dk} = \frac{d^3k}{(2\pi)^3 2k^0}\Big|_{k^0=\omega(\vec{k})}$$

is the Lorentz-invariant integration measure on the mass hyperboloid H_m.

This is the simplest possible quantum field. One can compute the commutation relations at different space-time points, and finds

$$[\varphi(x), \varphi(y)] = \int \widetilde{dk} \left(e^{-ik(x-y)} - e^{-ik(y-x)}\right) = (2\pi)^{-3} \int d^4k \, \mathrm{sign}(k^0)\, \delta(k^2 - m^2)\, e^{-ik(x-y)}.$$

This distribution vanishes at equal times when $\vec{x} \neq \vec{y}$, and because it is Lorentz invariant, it vanishes whenever x and y are spacelike separated, ie, the field is indeed a local field.

2.4.2 Fock representation

The postulated momentum space CCR can be realized as creation and annihilation operators on the Fock space, which is spanned by the (improper) vectors

$$a^*(k_1)\cdots a^*(k_n)\Omega$$

(symmetric under permutations). The action of $a(k)$ on such vectors is computed by commuting $a(k)$ past all $a^*(k_i)$, by using the momentum space CCR and the defining property $a(k)\Omega = 0$ of the ground state, as for the QM harmonic oscillator. Likewise, inner products between Fock space vectors are computed by using that the adjoint of $a^*(k)$ is $a(k)$, and proceeding as before. From these rules, it turns out that the Fock space has the structure

$$\mathcal{F}(\mathcal{H}_1) = \bigoplus_n \mathcal{H}_1^{\otimes n}\big|_{\rm symm},$$

where $\mathcal{H}_1$ is spanned by the improper vectors $|k\rangle = a^*(k)\Omega$, and $\mathcal{H}_1^{\otimes 0} \equiv \mathbb{C}\cdot\Omega$.

$\mathcal{H}_1$ is an irreducible repn of the Poincaré group, which induces a representation on the entire Fock space. The generators can be written as

$$P_\mu = \int \widetilde{dk}\, k_\mu a^*(k)a(k),$$

(and similar for the Lorentz generators $M_{\mu\nu}$). It follows that state vectors $a^*(k_1)\cdots a^*(k_n)\Omega$ are eigenvectors of P_μ with four momentum $k_1 + \cdots + k_n$.

Thus, the free field φ is defined on the Fock space. We can compute its 2-point function $W_2^{(m)}(x,y) \equiv \big(\Omega, \varphi(x)\varphi(y)\Omega\big)$:

$$W_2^{(m)}(x,y) = \int \widetilde{dk}\, e^{-ik(x-y)} = (2\pi)^{-3}\int d^4k\, \theta(k^0)\,\delta(k^2-m^2)\, e^{-ik(x-y)},$$

For $m = 0$, this can be displayed directly in x-space:

$$W_2^{(m=0)}(x,y) = \lim_{\varepsilon\downarrow 0} \frac{-(2\pi)^{-2}}{(x^0-i\varepsilon)^2 - \vec{x}^2} \equiv \lim_{\varepsilon\downarrow 0} \frac{-(2\pi)^{-2}}{x^2 - i\varepsilon{\rm sign}\,(x^0)}.$$

This "function" is actually a distribution, singular at $(x-y)^2 = 0$. The indicated "$i\varepsilon$-prescription" arises from the fact that the Fourier transform is supported on 4-momenta in the forward hyperboloid.

It is now easy to see that $W_2(x,x)$ is a divergent integral. This means that $\varphi(x)^2$ has a divergent vacuum expectation value (or: φ has infinite fluctuations in the vacuum state). Thus, quantum fields are too singular to be multiplied with each other, they are rather distributions. However, for free fields, one can define Wick products such as

$$:\varphi(x)^2:$$

where the Wick product $:\cdot:$ stands for "normal ordering", i.e., all annihilation operators a are moved to the right of creation operators a^*. Wick ordered products of free fields define new local quantum fields.

The most important example is the stress-energy tensor which is the symmetric tensor field, constructed in analogy with classical field theory,

$$\Theta_{\mu\nu}(x) = :\Big(\partial_\mu\varphi\partial_\nu\varphi - \eta_{\mu\nu}\big(\frac{1}{2}\,\partial_\kappa\varphi\partial^\kappa\varphi - \frac{m^2}{2}\,\varphi^2\big)\Big):$$

It satisfies the conservation laws $\partial_\kappa\Theta^{\mu\kappa}(x) = 0$ and $\partial_\kappa(x^\mu\Theta^{\nu\kappa}(x) - x^\nu\Theta^{\mu\kappa}(x)) = 0$. The corresponding conserved quantities are the generators of the Poincaré group:

$$P^\mu = \int d^3x\,\Theta^{\mu 0}(t,\vec{x}), \qquad M^{\mu\nu} = \int d^3x\,\big(x^\mu\Theta^{\nu 0}(t,\vec{x}) - x^\nu\Theta^{\mu 0}(t,\vec{x})\big),$$

where the integrals do not depend on the fixed time t. In particular, its components $T^{\mu 0}$ are the energy and momentum *densities*. These properties characterize the stress-energy tensor: in a general QFT, the SET is a symmetric conserved tensor field $T^{\mu\nu}$, such that its moments as above are the generators of the Poincaré group.

The SET can be used to explore local features of Hilbert space vectors like $\Phi = a^*(k_1)\cdots a^*(k_n)\Omega$ (in the free field case), such as their distribution of energy and momentum in space and time. It turns out that expectation values of the energy and momentum densities $\rho^\mu(x) = (\Phi, \Theta^{\mu 0}(x)\Phi)$ in such states are functions in space and time that are concentrated along trajectories with velocity k_i/m. This a posteriori justifies the interpretation of these states as multi-particle states.

We have found that the state space of a quantum *field* is spanned by states describing *particles*. This is the final meaning of the "particle–wave dualism", emerging naturally in quantum field theory. In the same way, the *states* of the electromagnetic field will describe photons, and electrons and positrons will arise as states of the Dirac *field*.

2.5 Correlations

Returning to the general case: one defines the correlation functions as the vacuum expectation values

$$W_n(x_1,\cdots,x_n) := (\Omega, \phi(x_1)\cdots\phi(x_n)\Omega)$$

(or more generally mixed correlations among different fields).

The reconstruction theorem states that one can reconstruct the fields and the Hilbert space from the knowledge of all correlation functions. (For experts: the reconstruction is a variant of the GNS construction, which in turn generalizes the reconstruction of a probability distribution when the moments of a random variable are known.) The correlation functions must satisfy a certain positivity

property, in order to ensure that the reconstructed Hilbert space has a positive-definite inner product. Otherwise, the corresponding "quantum theory" would not have a probabilistic interpretation, and would be essentially meaningless.

Moreover, when the correlation functions are invariant under joint Poincaré transformations:

$$W_n(x_1, \cdots, x_n) = W_n(\Lambda x_1 + a, \cdots, \Lambda x_n + a),$$

then the reconstructed Hilbert space is equipped with a unitary representation $U(a, \Lambda)$. A certain support property of the Fourier transforms of the correlation functions ensures that this representation has positive energy. (Another formulation in position space is to require that a correlation function is a distribution, which is the boundary value of an analytic function, similar as the "$i\varepsilon$-prescription displayed for the massless 2-point function above.)

Finally, when the correlation function is invariant under permutation of two neighboring arguments x_i and x_{i+1} provided $(x_i - x_{i+1})^2 < 0$, then the reconstructed quantum field is local.

Thus, the entire information about a quantum field is encoded in its correlation functions. In principle, one could construct a QFT by providing a formula how to produce arbitrary correlation functions. (One such formula consists in trying to define the correlation function by some path integral.)

However, as we have explained, the admissible correlation functions must satisfy a list of properties, which are not easy to fulfill simultaneously. One would therefore like to have a general understanding what the "most general form" of an admissible correlation function could be. Unfortunately, this has been achieved only for the case of 2-point functions ($n = 2$): The Källen-Lehmann representation of the most general scalar 2-point function is

$$W_2(x, y) = \int_{m \geq 0} d\rho(m^2)\, W_2^{(m)}(x, y),$$

where $W_2^{(m)}(x, y)$ is the 2-point function of the massive free field, as above, and $d\rho(m^2)$ is any positive, polynomially bounded measure.

2.5.1 Wick's theorem

Because free fields and their Wick products are expressed in terms of creation and annihilation operators, their correlation functions = vacuum expectation values can be computed by using their commutation relations, just as with the harmonic oscillator in QM. In practice, this is quite tedious. Wick's theorem states how this can be done almost without effort.

One needs to know the free 2-point function $W_2(x, y) = \big(\Omega, \varphi(x)\varphi(y)\Omega\big)$. Then a product

$$\varphi(x_1) \cdots \varphi(x_N)$$

can be written as a sum over all possible "contraction schemes", i.e., one selects pairs $i < j$ and replaces the pair of factors in the product by $W_2(x_i, x_j)$. The remaining uncontracted field operators are normal-ordered, and finally all such schemes are summed over. E.g.,

$$\varphi(x) = :\varphi(x):, \qquad \varphi(x_1)\varphi(x_2) = W_2(x_1,x_2) + :\varphi(x_1)\varphi(x_2):,$$

$$\begin{aligned}\varphi(x_1)\varphi(x_2)\varphi(x_3) &= W_2(x_1,x_2):\varphi(x_3): + W_2(x_1,x_3):\varphi(x_2): + W_2(x_2,x_3):\varphi(x_1):\\ &+ :\varphi(x_1)\varphi(x_2)\varphi(x_3):,\end{aligned}$$

while for $n \geq 4$, one also has terms with more than one contraction, such as

$$\begin{aligned}\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4) &= W_2(x_1,x_2)W_2(x_3,x_4) + W_2(x_1,x_3)W_2(x_2,x_4) +\\ &+ W_2(x_1,x_4)W_2(x_2,x_3)\\ &+ W_2(x_1,x_2):\varphi(x_3)\varphi(x_4): + \text{five similar terms}\\ &+ :\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4):.\end{aligned}$$

Since the vacuum expectation value of any normal ordered product with at least one factor is zero, the VEV of a product of an even number of field operators is just the sum over all fully contracted terms (n contractions), while the VEV of an odd number of field operators vanishes, because there is always at least one uncontracted field in the sum over contraction schemes. This yields all correlation functions of a free field as sums over products of 2-point functions. (This property of a field is actually equivalent to being a free field.)

To compute also correlation functions of Wick products of free fields, the following rule pertains: A product of Wick products

$$\big(:\varphi(x_1)\cdots\varphi(x_{n_1}):\big)\cdot\big(:\varphi(x_{n_1+1})\cdots\varphi(x_{n_2}):\big)\cdots\big(:\varphi(x_{n_r+1})\cdots\varphi(x_N):\big)$$

is obtained from the previous expansion for the product $\varphi(x_1)\cdots\varphi(x_N)$ by simply omitting all contraction schemes which involve a contraction of two fields which belong to the same Wick product of the left-hand side. E.g.,

$$\varphi(x_1)\cdot:\varphi(x_2)\varphi(x_3): = W_2(x_1,x_2):\varphi(x_3): + W_2(x_1,x_3):\varphi(x_2): + :\varphi(x_1)\varphi(x_2)\varphi(x_3):.$$

In this expression, one may equate the arguments of the fields within each Wick product of the left-hand side, so that, e.g.,

$$\varphi(x_1)\cdot:\varphi(x_2)^2: = 2\,W_2(x_1,x_2):\varphi(x_2): + :\varphi(x_1)\varphi(x_2)^2:.$$

Observe the appearance of combinatorial factors (like "2") in this expansion!

2.6 Mathematical nature of quantum fields

The 2-point function of the free field is

$$W_2(x,y) = \int \widetilde{dk}\, e^{-ik(x-y)} \equiv (2\pi)^{-3} \int d^4k\, \delta(k^2-m^2)\theta(k^0)\, e^{-ik(x-y)}.$$

At $x = y$, this is a divergent integral (actually, also at $(x-y)^2 = 0$). Notice that $W_2(x,x) = (\Omega, \phi(x)^2\Omega)$, if it were finite, would measure the statistical fluctuation of the operator $\phi(x)$ in the vacuum state. That it diverges tells us that the operator has infinite fluctations already in the simplest of all possible states! This exemplifies the general fact that quantum fields are necessarily extremely singular objects: their proper mathematical nature is that of an "operator-valued distribution", i.e., "smeared" operators

$$\varphi(f) := \int d^4x\, f(x)\, \varphi(x)$$

are much better behaved when f is a sufficiently smooth and fast decaying function: they are now unbounded operators defined on a dense domain of the Hilbert space containing the vacuum vector, and they can be multiplied with each other. In particular, they have finite fluctuations (depending on the smearing function f).

These properties are formulated in the Wightman axiomatic approach to QFT.

2.7 LSZ formula

In experiments, one usually measures cross sections in scattering processes. How do these relate to the correlation functions (given our claim, that the latter contain every theoretical information about the theory)?

The LSZ formula (Lehmann-Symanzik-Zimmermann) is the result of an analysis of the large-time behaviour of interacting quantum fields. It expresses differential scattering cross sections as squares of scattering amplitudes. The latter can be extracted from the Green functions of the theory: more precisely, they are certain residues of the Fourier transforms of the Green functions, which have poles in momentum space.

The Green functions in turn are vacuum expectation values of time-ordered field products. Time ordering means that a product of factors $\phi(x)$ is re-ordered according to the rule "later to the left":

$$T\big[\phi(x_1)\cdots\phi(x_N)\big] := \phi(x_{\pi(1)})\cdots\phi(x_{\pi(N)})$$

where π is a permutation such that $x^0_{\pi(1)} > \cdots > x^0_{\pi(N)}$. (This prescription seems not to be Lorentz invariant, but because of local commutativity, it is: the sign of the relative time can only be switched by a Lorentz transformation, when the two points are at spacelike separation. But in this case, the order of the operators does not matter.)

The time-ordered correlation functions = Green functions are obtained from the proper correlation functions just by applying the proper permutation of arguments, e.g.,

$$\big(\Omega, T\big[\phi_0(x)\phi_0(y)\big]\Omega\big) = \theta(x^0-y^0)W_2(x,y) + \theta(y^0-x^0)W_2(y,x).$$

2.8 Perturbation theory

In most practical approaches to interacting QFT, one specifies the interaction by a Lagrangean density $\mathcal{L}_I$ which typically is a local polynomial in the fields, e.g., the Standard Model). How can one compute the Green functions with this interaction?

The Gell-Mann–Low formula gives a formal computational recipe:

$$\big(\Omega, T\big[\phi(x_1)\cdots\phi(x_N)\big]\Omega\big) = \frac{(\Omega, T\big[\phi_0(x_1)\cdots\phi_0(x_N)e^{i\int d^4y :\mathcal{L}_I(\phi_0(y)):}\big]\Omega)}{(\Omega, T\big[e^{i\int d^4y :\mathcal{L}_I(\phi_0(y)):}\big]\Omega)}.$$

Here, ϕ on the left-hand side is the desired interacting field, while ϕ_0 on the right-hand side is a free field. Therefore, the right-hand side can be computed in terms of free field theory. In perturbation theory, one expands the exponential into a power series, so that the problem amounts to compute all expressions of the form

$$\frac{i^n}{n!}\big(\Omega, T\big[\phi_0(x_1)\cdots\phi_0(x_N):\mathcal{L}_I(\phi_0(y_1)):\cdots:\mathcal{L}_I(\phi_0(y_n)):\big]\Omega\big),$$

integrated over $y_1,\dots,y_n$ and summed over n. Thus, one needs to compute vacuum expectation values of time-ordered products of Wick products in free field theory.

This is again done by another generalization of Wick's theorem: it states that, in comparison with the previous ordinary product of Wick products, the time-ordering is completely taken care of if one only replaces the 2-point function $W_2(x,y)$ by the time-ordered 2-point function (Green function)

$$-iG_2(x,y) = \big(\Omega, T\big[\phi_0(x)\phi_0(y)\big]\Omega\big) = \theta(x^0-y^0)W_2(x,y) + \theta(y^0-x^0)W_2(y,x).$$

(The factor $-i$ is conventional. G_2 is also known as the Feynman propagator G_F.)

The combinatorics of all the possible contraction schemes is most efficiently represented pictorially in terms of Feynman diagrams, whose vertices correspond to the above interaction points y_i $(i=1,\dots,n)$, and the connecting lines correspond to the contractions. Contractions involving the field points x_k $(k=1,\dots,N)$ are depicted as lines with one or both ends not attached to a vertex, and labelled x_k.

Thus, every diagram with as many vertices as the order n of the perturbative expansion of the exponential $e^{i\int\mathcal{L}_I}$ corresponds to one term in the Wick expansion of the time-ordered VEV, with combinatorial factors. It is a product of coupling constants (coefficients appearing in $\mathcal{L}_I$), and Feynman propagators whose arguments are either x_k or y_i, as the structure of the graphs tells. These have to be integrated over y_i, and summed over n.

It turns out that the integration over y_i is in general divergent, and that the "most divergent" diagrams are vacuum diagrams (diagrams without external lines). It also turns out that the numerator of the GML formula, which is also treated as a power series $1 + \cdots$, precisely cancels all vacuum subdiagrams. But there are more divergent integrals. To deal with them, one has to start the renormalization program, in order to make sense of these formal computations. Here, perturbative QFT becomes really difficult. The summation over n is usually never performed, partly because one cannot compute higher order diagrams anyway, partly because comparison with the experiment tends to show good agreement already at low orders. But the most serious problem with the sum over n is that it is known to be divergent! Non-perturbative QFT consists in attempts to circumvent this problem by avoiding the power series expansion of the exponential altogether, and define the right-hand side of the Gell-Mann–Low formula in a different way.

3 Conformal symmetry

Masses and other dimensionfull parameters (in units $\hbar = c = 1$) set an absolute scale, since quantities like energy or distance can be measured relative to this scale. A theory which has no absolute scale, should "look the same" if all lengths are scaled by the same factor λ. This invariance is called dilation invariance. (Quantities like momentum or fields will then scale with a different power of λ.) Scale-free quantum systems often exhibit even a larger symmetry: Conformal symmetry.

3.1 Conformal symmetry of spacetime ($D > 2$)

Conformal transformations are transformations of space and time which change the scale by a position-dependent, but direction-independent factor. In other words: lengths may be scaled, but angles are invariant. In $D > 2$ dimensions, all (orientation-preserving) conformal transformations are products of Poincaré transformations, dilations

$$x^\mu \mapsto \lambda\, x^\mu \qquad (\lambda \in \mathbb{R}_+),$$

and special conformal transformations

$$x^\mu \mapsto \frac{x^\mu - x^2\, b^\mu}{1 - 2bx + b^2\, x^2} \qquad (b \in \mathbb{R}^D).$$

The latter are singular at points where $1-2bx+b^2\, x^2 = 0$. One may, however, regard Minkowski spacetime M^D just as a chart of the four-dimensional Dirac manifold $\overline{M^D}$, whose complement has measure zero. Then the conformal transformations are perfectly regular on $\overline{M^D}$, and the mentioned singularities are just coordinate singularities.

A convenient description of the Dirac manifold is as follows. Consider the auxiliary space $\mathbb{R}^{2,D} \equiv \mathbb{R}^{D+2}$ equipped with the metric $\eta_{AB} = \mathrm{diag}(+, - \cdots -, +)$

$(A, B = 0, \ldots, D+1)$. Consider the cone of null vectors $X \neq 0$ such that $\eta_{AB} X^A X^B = 0$. A ray in the cone is an equivalence class $X \sim \lambda X$ ($\lambda \in \mathbb{R} \setminus \{0\}$). The Dirac manifold $\overline{M^D}$ is the set of equivalence classes. Its elements can be regarded as pairs of two unit vectors $e = (X^0, X^{D+1}) \in \mathbb{R}^2$ and $f = (X^1, \ldots, X^D) \in \mathbb{R}^D$ with $(e, f) \sim (-e, -f)$, i.e.,

$$\overline{M^D} = \big(S^1 \times S^D\big)/\mathbb{Z}_2.$$

Since $SO(2, D)$ preserves the auxiliary metric, the conformal group is

$$\mathrm{Conf}_D = SO(2, D)_+/\mathbb{Z}_2,$$

where the subscript + denotes the orientation preserving subgroup.

The coordinates

$$x^\mu := \frac{X^\mu}{X^D + X^{D+1}} \qquad (\mu = 0, \ldots, D-1)$$

define the Minkowski chart for all points in $\overline{M^D}$ for which $X^D + X^{D+1} \neq 0$. Acting on $X \in \mathbb{R}^{2,D}$ with various subgroups of $SO(2, D)$, yields the conformal transformations of the Minkowski coordinates as above.

3.2 Conformal symmetry of quantum fields

Assuming conformal symmetry to be realized in quantum field theory, means replacing the Poincaré group in the previous chapter by the conformal group. Since fields also carry a dimension (they scale like some power d of mass), the scale transformation law reads

$$U(\lambda)\phi(x)U(\lambda)^* = \lambda^d\, \phi(\lambda x).$$

More generally, fields can be classified by the representation of Conf_D under which the states $\phi(x)\Omega$ transform. The irreducible unitary positive-energy representations of the conformal group in $D = 4$ dimensions are characterized by three parameters ("quantum numbers")

$$(d \geq 0; j_\pm = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots).$$

d is the (scaling) dimension, and (j_+, j_-) refers to the transformation law under the Poincaré group. Scalar fields have $j_+ = j_- = 0$, symmetric traceless tensor fields of rank r have $j_+ = j_- = \frac{1}{2}r$. The massless free scalar field has $(d; j_+, j_-) = (1; 0, 0)$, the massless free Dirac field (four components) splits in two chiral 2-spinors with $(d; j_+, j_-) = (\frac{3}{2}; \frac{1}{2}, 0)$ and $(\frac{3}{2}; 0, \frac{1}{2})$, and the six components of the free electromagnetic field $F_{\mu\nu}$ splits in two chiral parts $\vec{E} \pm \vec{B}$ with quantum numbers $(2; 1, 0)$ and $(2; 0, 1)$.

The transformation law under a general conformal transformation $g \in \mathrm{Conf}_D$ reads

$$U(g)\phi(x)U(g)^* = \big[\det(\partial(gx)^\mu/\partial x^\nu)\big]^{d/D}\, \phi(\lambda x).$$

for scalar fields of dimension d; the Jacobi determinant $\det(\partial(gx)^\mu/\partial x^\nu)$ must be replaced for general fields by a matrix representation acting on field multiplets.

The infinitesimal version of this law is (again only for scalar fields)

$$i[P_\mu,\phi(x)] = \partial_\mu\phi(x), \qquad i[M_{\mu\nu},\phi(x)] = \big(x_\mu\partial_\nu - x_\nu\partial_\mu\big)\phi(x),$$

$$i[D,\phi(x)] = \big(x^\kappa\partial_\kappa + d\big)\phi(x), \qquad i[K_\mu,\phi(x)] = \big(2x_\mu(x^\kappa\partial_\kappa) - x^2\partial_\mu + 2dx_\mu\big)\phi(x),$$

where D and K_μ are the infinitesimal generators for the dilations and for the special conformal transformations.

Like P^μ and $M^{\mu\nu}$, also the new generators D and K^μ are integrals over densities that can be expressed in terms of the stress-energy tensor. In the case of the massless free field, is convenient to add to the previous stress-energy tensor $\Theta_{\mu\nu}$ the term $-\frac{1}{3}{:}\big(\partial_\mu\varphi\partial_\nu\varphi - \varphi\partial_\mu\partial_\nu\varphi + \eta_{\mu\nu}\partial_\kappa\varphi\partial^\kappa\varphi\big){:}$. The total stress-energy tensor is then

$$T_{\mu\nu}(x) = {:}\Big(\frac{2}{3}\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{3}\varphi\partial_\mu\partial_\nu\varphi - \frac{1}{6}\eta_{\mu\nu}\,\partial_\kappa\varphi\partial^\kappa\varphi\Big){:}.$$

$T_{\mu\nu}$ satisfies the same conservation laws as $\Theta_{\mu\nu}$ before, and the generators P_μ and $M_{\mu\nu}$ do not change when Θ is replaced by T (because all integrals extend over gradients of fields). But $T_{\mu\nu}$ has the additional advantage that it is traceless:

$$T^\mu_\mu(x) \equiv \eta_{\mu\nu}T^{\mu\nu}(x) = 0.$$

This implies that the additional conservation laws hold: $\partial_\kappa(x_\mu T^{\mu\kappa}(x)) = 0$ and $\partial_\kappa\big(2x^\mu x_\nu T^{\nu\kappa}(x) - x^2T^{\mu\kappa}(x)\big) = 0$. The corresponding conserved quantities are the remaining generators of the conformal group:

$$D = \int d^3x\, x_\mu T^{\mu0}(t,\vec{x}), \qquad K^\mu = \int d^3x\,\big(2x^\mu x_\nu T^{\nu0}(t,\vec{x}) - x^2T^{\mu0}(t,\vec{x})\big),$$

where again the integrals do not depend on the fixed time t.

In general, the stress-energy tensor of a conformal QFT is a traceless symmetric conserved tensor field $T^{\mu\nu}$, such that its moments are the generators of the conformal group.

The "inversion" $I : x^\mu \mapsto \frac{x_\mu}{x^2}$ is a conformal transformation $I \in \mathrm{Conf}_D$ such that $I \circ T_a \circ I^{-1}$ is a special conformal transformation with $b^\mu = a_\mu$ when T_a is a translation by a_μ. Therefore, on the infinitesimal level, K^μ is unitarily conjugate to P_μ (by $U(I)$). It follows that, since P^0 has nonnegative spectrum, also K^0 has nonnegative spectrum. Then also the "conformal Hamiltonian"

$$L_0 = \frac{1}{2}\big(P^0 + K^0\big)$$

has nonnegative spectrum. L_0 is the infinitesimal generator for the "rotations" in the circle S^1 (in $\overline{M^D} = (S^1\times S^{D-1})/\mathbb{Z}_2$), therefore its spectrum is discrete. Indeed, in the representation associated with a field of scaling dimension d, the spectrum of L_0 is (contained in) the set $d + \mathbb{N}_0$.

In general in QFT, a classical symmetry may be represented only by a projective representation of the group (because states are state vectors up to a phase). For the Poincaré group this means, that one has to admit spinor representations in which space rotations by 2π are represented by $U(2\pi) = -\mathbf{1}$. The same is true for spinor representations of the conformal group ($j_+ + j_-$ = half-integer). In addition, when $d - j_+ - j_-$ is not an integer, the "rotations" in the S^1 factor of $\overline{M^D}$ are represented by a complex phase $U(2\pi) = e^{2\pi i L_0} = e^{2\pi i(d-j_+-j_-)}\mathbf{1}$.

(It is sometimes convenient to regard the fields to "live" on the universal covering space $\widetilde{M^D} = \mathbb{R} \times S^{D-1}$, so that the "rotation" in S^1 becomes a shift by 2π "into the next sheet".)

Conformal symmetry fixes the form of the 2-point function, except for a normalization. 2-point functions vanish unless the two fields have the same quantum numbers. For scalar and vector fields,

$$(\Omega, \phi_a(x_1)\phi_b(x_2)\Omega) \sim \frac{1}{\rho(x_{12})^d}, \qquad (\Omega, \phi^\mu_a(x_1)\phi^\nu_b(x_2)\Omega) \sim \frac{\eta^{\mu\nu}x_{12}^2 - 2x^\mu_{12}x^\nu_{12}}{\rho(x_{12})^{d+1}},$$

where $x_{12} \equiv x_1 - x_2$ and $\rho(x) \equiv -(x^0 - i\varepsilon)^2 + \vec{x}^{\,2}$. [1] It also fixes the form of scalar 3-point functions:

$$(\Omega, \phi_a(x_1)\phi_b(x_2)\phi_c(x_3)\Omega) \sim \frac{1}{\rho_{12}^{\frac{1}{2}(d_a+d_b-d_c)}\rho_{13}^{\frac{1}{2}(d_a-d_b+d_c)}\rho_{23}^{\frac{1}{2}(-d_a+d_b+d_c)}},$$

where $\rho_{ij} \equiv \rho(x_{ij})$. Scalar 4-point functions are determined up to a function of two "conformal cross ratios"

$$s = \frac{\rho_{12}\rho_{34}}{\rho_{13}\rho_{24}}, \qquad t = \frac{\rho_{14}\rho_{23}}{\rho_{13}\rho_{23}}.$$

Notice that cross ratios are invariant under conformal transformations.

The condition that the 2-point function describes a positive-definite scalar product between vectors $\phi(x)\Omega$, puts a lower bound on the possible scaling dimensions ("unitarity bound"), e.g., apart from the trivial field $\mathbf{1}$ with $d = 0$, one must have in $D = 4$ dimensions $d \geq 1$ for scalar fields, and $d \geq 2j + 2$ for symmetric traceless tensor fields ($j_+ = j_- = j > 0$) of rank $2j$.

We put $D = 4$ from now on.

3.3 Global Conformal Invariance (GCI)

Global Conformal Invariance is the postulate that covering representations do not occur. This implies that only integer d and integer $j_+ + j_-$ are admitted. In particular, one excludes perturbative approaches where the scaling dimension depends continuously on the coupling constant. Thus, GCI is a non-perturbative, axiomatic

[1] $(2\pi)^{-2} \cdot \rho(x_1 - x_2)^{-1}$ is just the free massless 2-point function, see the preceding chapter.

assumption: Correlations will *not* be computed by some GML formula, but rather classified by the general features they must satisfy.

GCI has strong consequences: apart from d and $j_+ + j_-$ being integers, it implies that all fields commute also at timelike distance (because there always is a conformal transformation, that maps a timelike pair to a spacelike pair). Thus non-cummutativity only occurs at lightlike separation: for this reason, this feature is called "Huygens property".

Moreover, all correlation functions are rational functions. More precisely, correlations of scalar fields of dimension d_i are of the general form

$$(\Omega, \phi_1(x_1)\cdots\phi_n(x_n)\Omega) = \sum_{\underline{\mu}} C_{\underline{\mu}} \prod_{i<j} \rho_{ij}^{\mu_{ij}},$$

where the sum extends over multi-indices $\underline{\mu} = (\mu_{ij})$, $\mu_{ij} = \mu_{ji} \in \mathbb{Z}$, subject to the "sum rule"

$$\sum_i \mu_{ij} = -d_i.$$

The sum rule ensures conformal invariance. Notice that the ratio of any two monomials for two admissible multi-indices can be written as a product of cross-ratios $\rho_{ij}\rho_{kl}/\rho_{ik}\rho_{jl}$; therefore, one may select any admissible multi-index $\underline{\mu}$, and write all other monomials

$$(\Omega, \phi_1(x_1)\cdots\phi_n(x_n)\Omega) = f(\text{cross ratios}) \cdot \prod_{i<j} \rho_{ij}^{\mu_{ij}},$$

where f is a sum of products of powers of cross ratios.

Correlation functions of tensor fields contain additional factors polynomial in the coordinate differences (and the powers of ρ_{ij} satisfy different sum rules).

3.4 Partial waves

The full Hilbert space $\mathcal{H}$ carries a positive-energy representation of the conformal group, hence of the Lie algebra

$$\begin{aligned}
[P_\mu, P_\nu] = [K_\mu, K_\nu] = [D, M_{\mu\nu}] = 0, \qquad & [D, P_\mu] = -iP_\mu, \qquad [D, K_\mu] = iK_\mu, \\
[M_{\mu\nu}, P_\kappa] = i(\eta_{\nu\kappa}P_\mu - \eta_{\mu\kappa}P_\nu), \qquad & [M_{\mu\nu}, K_\kappa] = i(\eta_{\nu\kappa}K_\mu - \eta_{\mu\kappa}K_\nu) \\
[M_{\mu\nu}, M_{\kappa\lambda}] = i(\eta_{\nu\kappa}M_{\mu\lambda} \mp \dots), \qquad & [P_\mu, K_\nu] = 2i\,\eta_{\mu\nu}D - 2i\,M_{\mu\nu}
\end{aligned}$$

of its generators. This Lie algebra possesses three "Casimir operators": polynomials in the generators that commute with all generators. The simplest one is quadratic:

$$C_2 = \frac{1}{2}\big(K_\mu P^\mu + P_\mu K^\mu\big) - D^2 + \frac{1}{2}M_{\mu\nu}M^{\mu\nu}.$$

By Schur's Lemma, every irreducible representation space is an eigenspace of the Casimir operators. One can therefore identify subrepresentations in $\mathcal{H}$ by diagonalizing the Casimir operators.

Consider an (improper) vector $\phi_1(x_1)\phi_2(x_2)\Omega$. Applying a Casimir operator, commuting it across the fields using the known commutation relations among generators $X = P, M, D, K$ and fields, and using the invariance of the vacuum $X\Omega = 0$, one finds a differential operator, e.g. (for $d_1 = d_2 = d$)

$$C_1\ \phi_1(x_1)\phi_2(x_2)\Omega = \rho_{12}^{-d}\big(x_{12}^2\,\partial_1\cdot\partial_2 - 2(x_{12}\otimes x_{12})\cdot(\partial_1\otimes\partial_2)\big)\rho_{12}^{-d}\cdot\phi_1(x_1)\phi_2(x_2)\Omega.$$

This implies relations of the form

$$(\Omega, \phi_1\cdots\phi_k C\phi_{k+1}\cdots\phi_n\Omega) = D(\Omega, \phi_1\cdots\phi_n\Omega),$$

where C is a Casimir operator (acting on the Hilbert space) and D the corresponding differential operator in the arguments $x_{k+1},\dots,x_n$ (or $x_1,\dots,x_k$) (acting on the correlation function). Because the Casimir operators have only certain (known) eigenvalues in positive-energy representations, every correlation function must be a sum of functions which are simultaneous eigenfunctions of these differential operators with admissible eigenvalues. These eigenfunctions can be thought of as contributions, called "partial waves",

$$(\Omega, \phi_1\Pi_{\lambda_1}\phi_2\Pi_{\lambda_2}\cdots\Pi_{\lambda_{n-1}}\phi_n\Omega)$$

where Π_λ are the projections on the subspaces of $\mathcal{H}$ corresponding to the representation λ. Because $\sum_\lambda \Pi_\lambda = \mathbf{1}$, the full correlation is a sum of the partial waves. (Actually, the projections Π_{n-1} and Π_1 are redundant, because vectors $\phi(x)\Omega$ belong to the representation determined by the quantum numbers of the field ϕ.)

The partial waves are therefore determined (up to a normalization) by a set of eigenvalue differential equations. The fact that the full correlation function can be represented as a linear combination of these eigenfunctions, poses further restrictions on the correlation functions. In particular, the admissible eigenvalues are only compatible with a maximal degree of singularity at lightlike distances $\rho_{ij} = 0$. This implies certain lower bounds for the admissible powers μ_{ij}.

These "pole bounds" in turn restrict the admissible multi-indices $\underline{\mu}$ to a finite set, hence every (scalar) correlation contains only a finite number of undetermined coefficients $C_{\underline{\mu}}$ (as in the preceding section). This is an enormous progress concerning the most general admissible form of correlations functions, as compared to the Poincaré invariant situation: a GCI QFT can be "parameterized" by a countable set of coefficients!

Yet, the coefficients $C_{\underline{\mu}}$ are subject to further constraints, to be discussed in the next section. It one could solve these constraints, one would have a complete classification of GCI quantum field theories. This ultimate goal is certainly out of reach, but one can make partial progress. The evaluation of the constraints on 4-point functions shows that some sets of coefficients are admitted that cannot be produced by correlation functions of Wick products of free fields, indicating the possible existence of unknown non-trivial (non-free) models.

3.5 Hilbert space positivity

Consider, for simplicity, a hermitean free field ϕ and the vector $\Phi_f = \int d^4x\, f(x)\phi(x)\Omega$. One can compute the norm of this vector:

$$\|\Phi_f\|^2 = \int d^4x d^4y\, \overline{f(x)} f(y)\, W_2(x,y).$$

That this quantity must be ≥ 0 for all f, is a (nonlinear) condition on the 2-point function. Indeed, the above unitarity bound on the scaling dimensions of fields arises from this condition: for dimensions below the unitarity bound, one can always find functions f_1 and f_2 such that $\|\Phi_{f_1}\|^2 > 0$ and $\|\Phi_{f_1}\|^2 < 0$ (i.e., changing the sign of W_2 does not help).

The same type of argument holds for vectors of the form $\int d^4x_1 d^4x_2\, f(x_1,x_2)$ $\phi_1(x_1)\phi_2(x_2)\Omega$, giving rise to a positivity condition on the 4-point function. Evaluating this condition by brute force "for all functions" $f(x_1,x_2)$ is very hard.[2]

The trick is, instead, to insert a projection as before, and require positive norm square for vectors of the form $\int d^4x_1 d^4x_2\, f(x_1,x_2)\ \Pi_\lambda \phi_1(x_1)\phi_2(x_2)\Omega$, for each λ separately. Their norm squares are integrals over the corresponding 4-point partial wave. Such partial waves are either positive or negative definite, so positivity of the 4-point function reduces to the determination of a sign, one for each representation λ. This method works if the partial waves are explicitly known. Unfortunately, they are not known for more that four points, and also not for tensor fields.

The "operator product expansion" says that vectors of the form form $\int d^4x_1 d^4x_2$ $f(x_1,x_2)\ \Pi_\lambda \phi_1(x_1)\phi_2(x_2)\Omega$ can also be written as $\int K_f(x)\ \phi_\lambda(x)\Omega$, where ϕ_λ is another quantum field with the transformation behaviour specified by the representation λ. (If ϕ_1 and ϕ_2 are free fields, then the fields ϕ_λ would just be certain Wick products of free fields.) In general, one only knows the existence of such fields.

Recent progress shows that one can directly obtain $\phi_\lambda(x)\Omega$ by application of a suitable differential operator D_λ in x_1 and x_2 to the vector $\phi_1(x_1)\phi_2(x_2)\Omega$, and then equating $x_1 = x_2 = x$. These differential operators are taylored such that they will annihilate all contributions from different representations $\lambda' \neq \lambda$, i.e., they have a similar effect as the insertion of a projection operator Π_λ.

Doing the same operation, say, on the first pair and the last pair of a $2n$-point correlation, reduces the latter to a $2n-2$-point correlation (now involving the new fields ϕ_λ). This must still be positive. Proceeding iteratively, one ends up with 2-point correlations for which positivity is just ensured by the correct sign of the coefficient.

In this way, one has an efficient test of positivity of $2n$-point functions: namely all coefficients obtained this way (with different representations λ chosen in each

[2]Indeed, in general QFT, positivity analysis is the most difficult part of the construction. It is very easy to construct correlation functions that have all desired properties except positivity, but to decide whether these are positive or not, is in general impossible.

step) must be positive. This is clearly only a necessary condition, but it can be developped to become also a sufficient condition.

If the given correlation functions are parametrized by coefficients $C_{\underline{\mu}}$, this amounts to finding a system of inequalities that these coefficients must satisfy – in general both upper and lower bounds.

This program has been accomplished for 4-point functions of scalar fields. Refinements concerning Cauchy-Schwartz inequalities rather than norm positivity are only partially under control. For tensor fields, the method is known to work "in principle", but has not been carried out in practice.

3.6 Conformal QFT in 2 dimensions

In two spacetime dimensions, the conformal group (the angle-preserving diffeomorphisms of spacetime $\mathbb{R}^2$) is much larger than $\mathrm{Conf}_2 = SO(2,2)_+/\mathbb{Z}_2$. Namely, because the metric can be written as $ds^2 = dx^{02} - dx^{12} = d(x^0 + x^1)d(x^0 - x^1)$, it contains arbitrary diffeomorphisms of the "chiral coordinates" $x^0 \pm x^1$ of the lightlike axes.

Yet, it turns out that 2-point functions are not invariant under these diffeomorphisms, but only under the group $\mathrm{Conf}_2 = SO(2,2)_+/\mathbb{Z}_2$. The diffeomorphism symmetry is broken because the vacuum vector is not invariant (see below).

The 2D Dirac manifold is, as in Sect. 3.1,

$$\overline{M^2} = (S^1 \times S^1)/\mathbb{Z}_2.$$

Introducing angular variables τ and ξ for the "timelike" and "spacelike" circles, the formula $x^\mu = \frac{X^\mu}{X^D + X^{D+1}}$ in Sect. 3.1 specializes to

$$x^\pm = x^0 \pm x^1 = \frac{\sin\tau}{\cos\tau + \cos\xi} \pm \frac{\sin\xi}{\cos\tau + \cos\xi} = \tan\frac{\tau \pm \xi}{2}.$$

It is also convenient to introduce

$$e^{i(\tau\pm\xi)} =: z^\pm = \frac{1 + ix^\pm}{1 - ix^\pm} \qquad \Leftrightarrow \qquad x^\pm = \frac{z^\pm - 1}{i(z^\pm + 1)}.$$

In these variables,

$$\overline{M^2} = S^1 \times S^1$$

with coordinates (z^+, z^-), a pair of complex numbers of modulus one. 2D Minkowski space is the chart $\mathbb{R} \times \mathbb{R}$ with coordinates (x^+, x^-).

The conformal group can be decomposed as

$$\mathrm{Conf}_2 = \mathrm{M\ddot{o}b}_+ \times \mathrm{M\ddot{o}b}_-,$$

where $\mathrm{M\ddot{o}b} = SL(2)/\mathbb{Z}_2$ is the group of fractional linear transformations

$$gx = \frac{ax + b}{cx + d},$$

and $\mathrm{M\ddot{o}b}_\pm$ acts on $x^\pm$ and leaves $x^\mp$ invariant.

The one-parameter subgroups $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} e^{\frac{1}{2}t} & 0 \\ 0 & e^{-\frac{1}{2}t} \end{pmatrix}$, and $\begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}$, respectively, correspond to the translations $x \to x + a$, dilations $x \to \lambda x$ and special conformal transformations $x \to x/(1 - bx)$.

Since quantum correlation functions are boundary values of analytic functions, one may study the analytically continued correlation functions at the "Euclidean points" where x^0 is purely imaginary, hence $(x^+ = iy + x, x^- = iy - x)$ are no longer two independent real numbers, but instead a complex conjugate pair $(\zeta, -\overline{\zeta})$; likewise (z^+, z^-) are no longer independent complex numbers of modulus one, but instead a complex conjugate pair $(z, \overline{z})$ in the complex plane. These "Schwinger functions" describe correlations in a classical field theory in two Euclidean space dimensions (no time), such as correlations of a Statistical Mechanics system on a surface, at thermal equilibrium. In such systems, conformal invariance is an emergent feature at critical points. Most of the formulae are more or less the same in Euclidean CFT and in Minkowski spacetime conformal QFT (except for the Cayley transform $x \mapsto z$), but the physical interpretation is quite different. The most notable difference is that in Euclidean CFT there is no need for a Hilbert space with positive metric, so that notions related to unitarity are either absent or strongly relaxed.

The SM interpretation of 2D CFT has also strongly influenced the terminology and notations. In order to emphasize that conformal QFT is just a special case of general QFT with additional symmetries, I prefer to retain QFT terminology and notations, which may cause some pains when comparing with CFT literature.

The generators of a unitary representation of Conf_2 can be rewritten as

$$P_\pm = \frac{1}{2}(P_0 \pm P_1), \quad D_\pm = \frac{1}{2}(D \pm M_{01}), \quad K_\pm = \frac{1}{2}(K_0 \mp K_1),$$

which satisfy the commutation relations of two commuting Lie algebras $sl(2, \mathbb{R})$ of the Möbius group

$$[D, P] = -iP, \quad [D, K] = iK, \quad [P, K] = 2iD.$$

Positivity of the spectrum of P^0 in every 2D Lorentz frame implies that P_+ and P_- must both be positive. Positive-energy representation of $sl(2, \mathbb{R})$ are uniquely determined by a real parameter $h \geq 0$ (the lowest eigenvalue of $L_0 = \frac{1}{2}(P + K)$), so that conformal covariant fields are characterized by a pair (h^+, h^-). The sum $h^+ + h^-$ is the scaling dimension, and the difference $h^+ - h^-$ is the helicity (the analog of spin in two dimensions). The infinitesimal transformation law of fields $\phi(x^+, x^-)$ read

$$i[P_\pm, \phi] = \partial_\pm \phi, \quad i[D_\pm, \phi] = (x^\pm \partial_\pm + h^\pm)\phi, \quad i[K_\pm, \phi] = (x^{\pm 2}\partial_\pm + 2x^\pm h^\pm)\phi.$$

This fixes 2- and 3-point correlations up to normalization:

$$(\Omega, \phi_1(x_1)\phi_2(x_2)\Omega) \sim (x^+_{12})^{-2h^+}_\varepsilon (x^-_{12})^{-2h^-}_\varepsilon$$

provided both fields have the same dimensions $h^\pm$ (otherwise $= 0$), and

$$(\Omega, \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\Omega) \sim (x^+_{12})^{-h^+_1-h^+_2+h^+_3}_\varepsilon (x^+_{13})^{-h^+_1+h^+_2-h^+_3}_\varepsilon (x^+_{23})^{+h^+_1-h^+_2-h^+_3}_\varepsilon \cdot (+\to -).$$

Here $(x)^{-\nu}_\varepsilon$ stands for the distribution $\lim_{\varepsilon\downarrow 0}\big(\frac{-i}{x-i\varepsilon}\big)^\nu = \Gamma(\nu)^{-1}\int_0^\infty k^{\nu-1}dk\, e^{-ikx}$.

The most remarkable feature of 2D conformal QFT is the presence of "chiral fields", with the stress-energy tensor as prime example: By symmetry $T^{\mu\nu} = T^{\nu\mu}$ and tracelessness $T^\mu_\mu = 0$, the SET has only two independent components $T^{00} = T^{11}$ and $T^{01} = T^{10}$. The conservation law $\partial_\mu T^{\mu\nu}$ implies $\partial_+(T^{00} - T^{01}) = 0$ and $\partial_-(T^{00} + T^{01}) = 0$, i.e., these combinations are fields that depend on a single chiral variable $x^\pm$

$$T_+(x^+) := T_{00}(x) + T_{01}(x) \qquad \text{and} \qquad T_-(x^-) := T_{00}(x) - T_{01}(x).$$

Because $T_+(x^+)$ can be expressed in terms of components of $T^{\mu\nu}(x')$ for any point x along a lightray with $x'^0 + x'^1 = x^+$, and similar for $T_-(y^-)$, it follows from locality that $T_+(x^+)$ commutes with $T_-(y^-)$, because there are always pairs of points along the two lightrays that are spacelike separated. The same holds true for $T_+(x^+)$ and $T_+(y^+)$, provided $x^+ \neq y^+$, and $(+\to -)$.

Thus, we have found two independent fields "defined on the lightray": fields of different chirality commute, and fields of equal chirality commute if their arguments differ. (In general, there may be further chiral fields sharing these properties.) Also the chiral generators of the Möbius group are moments of the chiral stress-energy tensors

$$P_\pm = \int T_\pm(x)\, dx, \qquad D_\pm = \int xT_\pm(x)\, dx, \qquad K_\pm = \int x^2 T_\pm(x)\, dx.$$

Chiral QFT (keeping only the chiral fields of one chirality) is a kinematically extremely simple type of QFT. It unites "in a single dimension" the spacelike feature of local commutativity with the timelike feature of positive spectrum of the time translations.

Chiral commutators can only be sums of derivatives of $\delta(x-y)$, multiplied by other fields. For the SET, this property, together with the requirement that its moments are the generators of the conformal group, uniquely determines the commutation relations

$$i[T(x), T(y)] = -\big(T(x) + T(y)\big)\delta'(x-y) + \frac{c}{24\pi}\delta'''(x-y)\mathbf{1},$$

up to an undetermined "central charge" $c > 0$. The central charge must be positive, because the commutation relation determines the 2-point function

$$(\Omega, T(x)T(y)\Omega) = \frac{c}{8\pi^2}(x-y)^{-4}_\varepsilon.$$

Consider the moments of the SET

$$L_n = L^*_{-n} = \frac{1}{2}\int (1+ix)^{1+n}(1-ix)^{1-n}\, T(x).$$

Clearly, $L_0 = \frac{1}{2}(P+K)$ and $L_{\pm 1} = \frac{1}{2}(P-K) \pm iD$ are linear combinations of the Möbius generators. The CR of the SET implies the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{n+m,0}.$$

This algebra may be regarded as the quantum version of the Lie algebra of the diffeomorphism group, where L_n are generators of infinitesimal diffeomorphisms that in the complex coordinate z on the circle are given by $\delta_n z \sim z^{n+1}$. The presence of the central charge means that the diffeomorphisms are represented only projectively.

Because the vacuum vector is Möbius invariant, it is annihilated by L_0 and $L_{\pm 1}$. Positivity of the energy requires nonnegative spectrum of L_0. Because $[L_n, L_0] = -nL_n$, generators with $n > 0$ lower the eigenvalue of L_0. Therefore one must have $L_n\Omega = 0$ $(n > 0)$. Then $\|L_{-n}\Omega\|^2 = (\Omega, L_n L_{-n}\Omega) = (\Omega, [L_n, L_{-n}]\Omega) = \frac{c}{12} n(n^2-1) > 0$ for $n > 1$, hence the vacuum vector is Möbius invariant, but not diffeomorphism invariant: this is why the larger symmetry of angle-preserving maps beyond $\mathrm{Conf}_2 = SO(2,2)_+/\mathbb{Z}_2 = \mathrm{M\ddot{o}b}_+ \times \mathrm{M\ddot{o}b}_-$ is necessarily broken.

A remarkable, highly nontrivial result states that positive-energy representations of this algebra exist only if c is one of the numbers

$$c = 1 - \frac{6}{m(m+1)} \quad (m = 3, 4, \dots) \qquad \text{or} \qquad c \geq 1.$$

Other values cannot occur in conformal QFT. This result was obtained by an algebraic approach to Hilbert space positivity, which is much more efficient than the approach via partial wave analysis as described in the previous section, but it works only for $c < 1$. It is even possible to classify and construct all conformal quantum field theories whose chiral stress-energy tensors have $c^+ = c^- < 1$: for each admissible $c < 1$, there is only a small number of such theories. A complete classification of quantum field theories of a certain type is an unprecendented achievement.

Notice that a full 2D conformal QFT will contain, apart from the SET, other fields: possibly there are more chiral field, but in addition there will be local fields that depend on both x^+ and x^-. The latter are called "primary" if their commutators with $T_\pm$ contain only the field and its derivatives, but no other fields. Commutators of primary fields with the SET are completely fixed by locality and conformal symmetry. When $c < 1$, this information can be exploited to obtain their correlation functions.

Partial waves are defined in the same way as in 4D. Since the conformal group splits into two chiral Möbius groups, the partial waves factorize into chiral partial waves, which are functions of x_i^+, resp. of x_i^- only. Since there is only one Casimir operator $C = \frac{1}{2}(PK+KP) - D^2$ for $sl(2,\mathbb{R})$, their determination is much easier.